Weakly nonlinear instability of a viscoelastic liquid jet

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Abstract
We present a temporal stability analysis of a viscoelastic liquid jet. Pure-deformation initial conditions are imposed and the viscoelastic liquid is represented by an Oldroyd-B model. The analysis is performed up to second-order, with the small parameter being the dimensionless initial deformation amplitude of the imposed mode. The jet problem depends on four other dimensionless numbers: an Ohnesorge number, two Deborah numbers built on the characteristic times of the fluid model and the dimensionless wavenumber of the imposed mode. The results are compared to the Newtonian case, recently treated in the literature, that is retrieved by the identity of the two Deborah numbers. It is found that the first-order dispersion relation admits an additional solution that is associated to a decaying mode. Three behaviours are predicted according to the position of the wavenumber with respect to the cut-off wavenumber $k_c = 1$, that is unchanged by viscoelasticity, and a critical-oscillation wavenumber k^* that depends more on the Ohnesorge number than on the Deborah numbers: growth without oscillations for $0 < k \leq k^*$, growth with oscillations for $k^* < k < k_c$ and damping with oscillations for $k \geq k_c$. Concerning the second-order solution, it is found that the Poisson equation for the second-order pressure admits an additional contribution containing products of modified Bessel functions with different arguments, that requires a polynomial approximation to be solved. The second-order solution is obtained by following the same method as the one used for the Newtonian case, except that three modes need to be considered instead of two in the Newtonian case.

Keywords
Jets, Nonlinear instability, Viscoelasticity

Introduction
Viscoelastic liquids are a common type of non-Newtonian fluids. These fluids behave like elastic solids at short times compared to the longest time of the fluid and like viscous liquids at longer times. Their elastic property, that is of scientific and industrial interest, is due to a macromolecular structure.
Macomolecular fluids are challenging because their motions cannot be simply described by the Navier-Stokes equations. The diversity of their structures, the molecular weight distributions and the large number of internal degrees of freedom make their molecular modelling very different from the one of Newtonian liquids. This leads to a large number of molecular models in the literature [1]. The most common ones consist in representing the macromolecules by chains of N springs. In particular, the N=2 case, referred to as the elastic dumbbells, is very convenient to use as it minimises mathematical complications and is often sufficient to predict macroscopic behaviours.
The theoretical description of a fluid flow is not only challenging when the fluid model is not Newtonian, but also when the geometry is not planar. These two difficulties have been put forward by Yarin in his seminal book [2]. In the case of a liquid jet, the cylindrical geometry adds a nonlinearity in the equations with respect to the spatial radial variable, generating Bessel functions in the solutions.
The weakly nonlinear stability analysis of a Newtonian jet was recently treated by Renoult et al. [3] following the work of Yuen on the inviscid case [4]. Here, we propose to extend the study to the case of a Hookean dumbbells viscoelastic fluid. The assumptions and notations of the problem are first given. The problem is then formulated in dimensional and dimensionless equations. The method of solution is briefly recalled as it is the same as the one deployed for the Newtonian case [3]. The equations and solutions are finally presented for the two first orders of the analysis. The article ends with a conclusion.

The formulation of the problem
We study the temporal stability of a viscoelastic liquid jet in a gaseous phase. The following assumptions are made:

- The jet is of infinite length.
- The jet is axisymmetric.
- The jet is initially at rest.
- The gas phase has negligible density and viscosity compared to the ones of the liquid phase.
- The liquid phase is incompressible and is represented by the Oldroyd-B rheological model.
- The surface tension between the two phases is constant.
- Gravity is not taken into account.

- The perturbation is varicose, of small-amplitude and single-mode.

The axisymmetric problem is described by cylindrical coordinates in the azimuthal symmetry plane $(O, \boldsymbol{e}_r, \boldsymbol{e}_z)$. The jet evolution is characterised by three space- and time-dependent quantities: $r_s(z,t)$ the jet surface position, $p(r,z,t)$ the liquid pressure and $\boldsymbol{v}(r,z,t) = u(r,z,t)\boldsymbol{e}_r + w(r,z,t)\boldsymbol{e}_z$ the liquid velocity field. The flow variables are parameterized by 8 constant parameters: a the radius of the undisturbed jet, ρ the liquid density, μ_0 the liquid zero-shear viscosity, λ_1 the liquid relaxation time, λ_2 the liquid deformation retardation time, σ the surface tension between the two phases, p_G the gas pressure, η_0 the small initial amplitude of the perturbation and k the wavenumber of the perturbation. Figure 1 presents a sketch of the jet configuration studied. Note that the mean radius of the disturbed jet does not equal a. It depends on η_0 and its value is derived by applying volume conservation between the undisturbed jet and the disturbed one at $t = 0$.

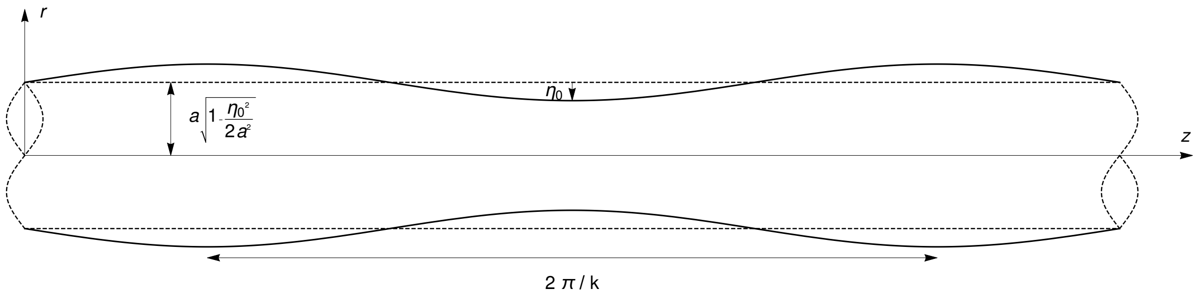

Figure 1. Sketch of the liquid jet under varicose deformation.

The governing equations

The problem in now formulated in a set of equations for the liquid phase. We distinguish the equations valid at all points in the liquid bulk, from the jump conditions valid only at the jet surface and from the initial conditions.
The liquid flow is governed by the mass and momentum balance equations. These equations read for $r \leq r_s(z,t)$, respectively:

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$$
$$\rho\left(\boldsymbol{v}_{,t} + \left(\boldsymbol{v} \cdot \boldsymbol{\nabla}\right)\boldsymbol{v}\right) = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\tau}$$

where $\boldsymbol{\tau}$ is the extra-stress tensor representing the liquid response to rates of deformation. Here, $\boldsymbol{\tau}$ is related to the rate-of-deformation tensor $\boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{\nabla v} + (\boldsymbol{\nabla v})^T\right)$ and the vorticity tensor $\boldsymbol{W} = \frac{1}{2}\left(\boldsymbol{\nabla v} - (\boldsymbol{\nabla v})^T\right)$ by the Oldroyd-B Rheological Constitutive Equation (RCE):

$$\boldsymbol{\tau} + \lambda_1 \overset{\circ}{\boldsymbol{\tau}} = 2\mu_0(\boldsymbol{D} + \lambda_2 \overset{\circ}{\boldsymbol{D}})$$

Note that viscoelasticity can be described by this model only if $\lambda_1 > \lambda_2$ and that for $\lambda_1 = \lambda_2$ the Newtonian case is retrieved. The "°" operator designates the upper-convected derivative specific to the Oldroyd-B model, which is defined for all tensors $\boldsymbol{A}$ by:

$$\overset{\circ}{\boldsymbol{A}} = \boldsymbol{A}_{,t} + \left(\boldsymbol{v} \cdot \boldsymbol{\nabla}\right)\boldsymbol{A} + \boldsymbol{W} \cdot \boldsymbol{A} - \boldsymbol{A} \cdot \boldsymbol{W} - \left(\boldsymbol{D} \cdot \boldsymbol{A} + \boldsymbol{A} \cdot \boldsymbol{D}\right)$$

The jump conditions correspond to the mass and momentum equations at the free surface. These equations read at $r = r_s$, respectively:

$$\rho(\boldsymbol{v} \cdot \boldsymbol{n} - \boldsymbol{v}_n) = 0$$
$$(p - p_G + \sigma\kappa)\boldsymbol{n} - \boldsymbol{\tau} \cdot \boldsymbol{n} = \boldsymbol{0}$$

where $\boldsymbol{n}(z,t)$ is the unit normal vector at the surface pointing outwards, $v_n(z,t)$ is the normal velocity of a surface point, $\kappa(z,t)$ is the local curvature of the surface. The expressions of these three quantities are respectively:

$$\boldsymbol{n}(z,t) = \frac{\boldsymbol{\nabla}(r - r_s)}{|\boldsymbol{\nabla}(r - r_s)|} = \frac{1}{\sqrt{1 + {r_{s,z}}^2}}\begin{pmatrix} 1 \\ 0 \\ -r_{s,z} \end{pmatrix}$$
$$\boldsymbol{v}_n(z,t) = \frac{-(r - r_s)_{,t}}{|\boldsymbol{\nabla}(r - r_s)|} = \frac{r_{s,t}}{\sqrt{1 + {r_{s,z}}^2}}$$
$$\kappa(z,t) = \frac{r_{s,zz}}{(1 + r_{s,z}^2)^{3/2}} - \frac{1}{r_s(1 + r_{s,z}^2)^{1/2}}$$

The jet is subject to pure-deformation initial conditions. These conditions read:

$$r_s(z,t=0) = \eta_0\cos(kz) + a\left[1 - \frac{1}{2}\left(\frac{\eta_0}{a}\right)^2\right]^{1/2}$$
$$r_{s,t}(z,t=0) = 0$$

Non-dimensional equations

The previous equations are non-dimensionalized using the undeformed jet radius a, the capillary time scale $(\rho a^3/\sigma)^{1/2}$ and the capillary pressure σ/a for length, time, and stress, respectively. The dimensionless governing equations of the problem are:

$$\frac{1}{r}(ru)_{,r} + w_{,z} = 0 \qquad \text{for} \quad r \le r_s$$
$$u_{,t} + uu_{,r} + wu_{,z} = -p_{,r} + \frac{1}{r}(r\tau_{rr})_{,r} - \frac{\tau_{\theta\theta}}{r} + \tau_{zr,z} \qquad \text{for} \quad r \le r_s$$
$$w_{,t} + uw_{,r} + ww_{,z} = -p_{,z} + \frac{1}{r}(r\tau_{rz})_{,r} + \tau_{zz,z} \qquad \text{for} \quad r \le r_s$$
$$\boldsymbol{\tau} + De_1\overset{\circ}{\boldsymbol{\tau}} = 2\,Oh_0(\boldsymbol{D} + De_2\overset{\circ}{\boldsymbol{D}}) \qquad \text{for} \quad r \le r_s$$
$$u = r_{s,t} + w\,r_{s,z} \qquad \text{at} \quad r = r_s$$
$$(\boldsymbol{\tau}\cdot\boldsymbol{n})\times\boldsymbol{n} = \boldsymbol{0} \qquad \text{at} \quad r = r_s$$
$$p + \kappa - (\boldsymbol{\tau}\cdot\boldsymbol{n})\cdot\boldsymbol{n} = 0 \qquad \text{at} \quad r = r_s$$
$$r_s(z,t=0) = \eta_0\cos(kz) + \left[1 - \frac{\eta_0^2}{2}\right]^{1/2}$$
$$r_{s,t}(z,t=0) = 0$$

Five dimensionless numbers appear in the above set of equations. The Ohnesorge number defined as $Oh_0 = \mu_0/(\sigma a\rho)^{1/2}$ weights the effect of the zero-shear viscosity. The two Deborah numbers De_1 and De_2 represent the dimensionless stress relaxation and deformation retardation times of the liquid, respectively. The last two dimensionless numbers are not related to the nature of the liquid but to the pure-deformation initial conditions. They are the dimensionless wavenumber k, and the dimensionless initial deformation amplitude η_0.

The method of solution

We apply a small-amplitude perturbation method to solve the problem. Here, the small parameter is the initial deformation amplitude η_0 and the development is performed up to second order. To that end, the physical quantities of the jet flow are expanded in power series with respect to η_0 up to second-order as follows:

$$u(r,z,t) = u_1(r,z,t)\,\eta_0 + u_2(r,z,t)\,\eta_0^2$$
$$w(r,z,t) = w_1(r,z,t)\,\eta_0 + w_2(r,z,t)\,\eta_0^2$$
$$p(r,z,t) = 1 + p_1(r,z,t)\,\eta_0 + p_2(r,z,t)\,\eta_0^2$$
$$\boldsymbol{\tau}(r,z,t) = \boldsymbol{\tau}_1(r,z,t)\,\eta_0 + \boldsymbol{\tau}_2(r,z,t)\,\eta_0^2$$
$$\boldsymbol{D}(r,z,t) = \boldsymbol{D}_1(r,z,t)\,\eta_0 + \boldsymbol{D}_2(r,z,t)\,\eta_0^2$$
$$\boldsymbol{W}(r,z,t) = \boldsymbol{W}_1(r,z,t)\,\eta_0 + \boldsymbol{W}_2(r,z,t)\,\eta_0^2$$
$$r_s(z,t) = 1 + \eta_1(z,t)\,\eta_0 + \eta_2(z,t)\,\eta_0^2$$

To minimise mathematical complications during the solution of the problem, the three equations representing the jump conditions that are satisfied on the deformed jet surface are rearranged using Taylor expansions, such as:

$$u(r=r_s,z,t) = u(r=1,z,t) + (r_s-1)\,u_{,r}(r=1,z,t) + ...$$
$$w(r=r_s,z,t) = w(r=1,z,t) + (r_s-1)\,w_{,r}(r=1,z,t) + ...$$
$$p(r=r_s,z,t) = p(r=1,z,t) + (r_s-1)\,p_{,r}(r=1,z,t) + ...$$

Since $\eta_0 \ll 1$ by assumption, the initial deformation can be approximated up to second order as follows:

$$r_s(z,t=0) = 1 + \eta_0\cos(kz) - \frac{1}{4}\eta_0^2$$

First-order equations and solutions

The first-order set of equations reads:

$$
\begin{aligned}
&\frac{1}{r}(ru_1)_{,r} + w_{1,z} = 0 && \text{for} \quad r \le r_s \\
&u_{1,t} = -p_{1,r} + \frac{1}{r}(r\tau_{rr1})_{,r} - \frac{\tau_{\theta\theta 1}}{r} + \tau_{zr1,z} && \text{for} \quad r \le r_s \\
&w_{1,t} = -p_{1,z} + \frac{1}{r}(r\tau_{rz1})_{,r} + \tau_{zz1,z} && \text{for} \quad r \le r_s \\
&\boldsymbol{\tau_1} + De_1\boldsymbol{\tau}_{1,t} = 2\,Oh_0\ (\boldsymbol{D}_1 + De_2\boldsymbol{D}_{1,t}) && \text{for} \quad r \le r_s \\
&u_1 = \eta_{1,t} && \text{at} \quad r = 1 \\
&\tau_{rz1} = 0 && \text{at} \quad r = 1 \\
&p_1 + \eta_1 + \eta_{1,zz} - \tau_{rr1} = 0 && \text{at} \quad r = 1 \\
&\eta_1(z, t = 0) = \cos(kz) \\
&\eta_{1,t}(z, t = 0) = 0
\end{aligned}
$$

The above equations are linear with respect to the time variable. We therefore search for complex solutions with an exponential time dependence under the form: $e^{-\alpha_1 t}$ where α_1 is the first-order mode eigenvalue of the instability. The real part of α_1 corresponds to the growth rate whereas the imaginary part corresponds to the oscillation frequency. Given that time dependency of the flow quantities, the first-order RCE becomes:

$$
\boldsymbol{\tau}_1 = 2\,\frac{1 - \alpha_1 De_2}{1 - \alpha_1 De_1}\,Oh_0\,\boldsymbol{D}_1 := 2\,\beta_1 Oh_0\,\boldsymbol{D}_1 := 2Oh_{v1}\,\boldsymbol{D}_1
$$

This result means that the first-order extra-stress tensor of the viscoelastic fluid model differs from the one of a Newtonian material only by a multiplying factor β_1 that depends on the two Deborah numbers and on the first-order mode, but not on the spatial variables. The method of solution of the linear problem is therefore formally the same as in the Newtonian case, treated by Renoult et al. [3], the Ohnesorge number Oh_0 being replaced by the modified Ohnesorge number Oh_{v1}. The first-order jet surface shape, velocity components and pressure thus read:

$$
\begin{aligned}
\eta_1(z,t) &= \Re e\left[\hat{\eta}_1 e^{ikz-\alpha_1 t}\right] \\
u_1(r,z,t) &= \Re e\left[\hat{\eta}_1\left(\left(2k^2 Oh_{v1} - \alpha_1\right)\frac{I_1(kr)}{I_1(k)} - 2Oh_{v1}k^2\frac{I_1(l_{v1}r)}{I_1(l_{v1})}\right)e^{ikz-\alpha_1 t}\right] \\
w_1(r,z,t) &= \Re e\left[i\hat{\eta}_1\left(\left(2k^2 Oh_{v1} - \alpha_1\right)\frac{I_0(kr)}{I_1(k)} - 2Oh_{v1}kl_v\frac{I_0(l_{v1}r)}{I_1(l_{v1})}\right)e^{ikz-\alpha_1 t}\right] \\
p_1(r,z,t) &= \Re e\left[\frac{\hat{\eta}_1\alpha_1}{k}\frac{I_0(kr)}{I_1(k)}\left(2k^2 Oh_{v1} - \alpha_1\right)e^{ikz-\alpha_1 t}\right]
\end{aligned}
$$

where $l_{v1}^2 := k^2 - \alpha_1/Oh_{v1}$ defines a first-order modified wavenumber. $\hat{\eta}_1$ is an amplitude parameter, I_0 and I_1 are the modified Bessel functions of the first kind of order 0 and 1, respectively.
And the first-order dispersion relation is given by:

$$
DR : \alpha_1^2 + \alpha_1\,B(\alpha_1, k, Oh_0, De_1, De_2) + C(\alpha_1, k, Oh_0, De_1, De_2) = 0
$$

$$
\begin{aligned}
\text{with} : B(\alpha_1, k, Oh_0, De_1, De_2) &= -2k^2 Oh_{v1}\left[1 - \frac{I_1(k)}{I_0(k)}\left(\frac{1}{k} + \frac{2kl_{v1}}{l_{v1}^2 + k^2}\left(\frac{I_1(l_{v1})}{I_0(l_{v1})} - \frac{1}{l_{v1}}\right)\right)\right] \\
C(\alpha_1, k, Oh_0, De_1, De_2) &= -k(1-k^2)\frac{I_1(k)}{I_0(k)}\frac{l_{v1}^2 - k^2}{l_{v1}^2 + k^2}
\end{aligned}
$$

which was first presented by Goldin et al. [5], then derived again and examined in more details by Brenn et al. [6]. For $De_1 = De_2$, the factor β_1 equals unity and this relation reduces to the dispersion relation of a Newtonian liquid first presented by Rayleigh [7].

The dispersion relation (DR) is solved for a given set of parameters (k, Oh_0, De_1, De_2) by determining the zeros of the transcendental DR function using the polynomial method described in [8]. This method is based on the Cauchy's integral theorem. It is found that the dispersion relation admits four distinct solutions: one identically zero solution $\alpha_1 = 0$ and three non-identically zero solutions. These latter solutions are continuous with respect to wavenumber k. One solution is of particular interest: its growth rate cancels at $k = 1$ whatever the values of the other three parameters. This solution is denoted α_{1d} and the two others α_{1sa} and α_{1sb}. It should be reminded that for the Newtonian case the dispersion relation admits only two solutions besides the trivial solution (that is generally omitted in the previous work). These two solutions correspond to the capillary modes reported in the work of García & Gonzáles [9]. The fact that an additional solution is found for the viscoelastic case is due to the eigenvalue-dependency of the modified Ohnesorge number.

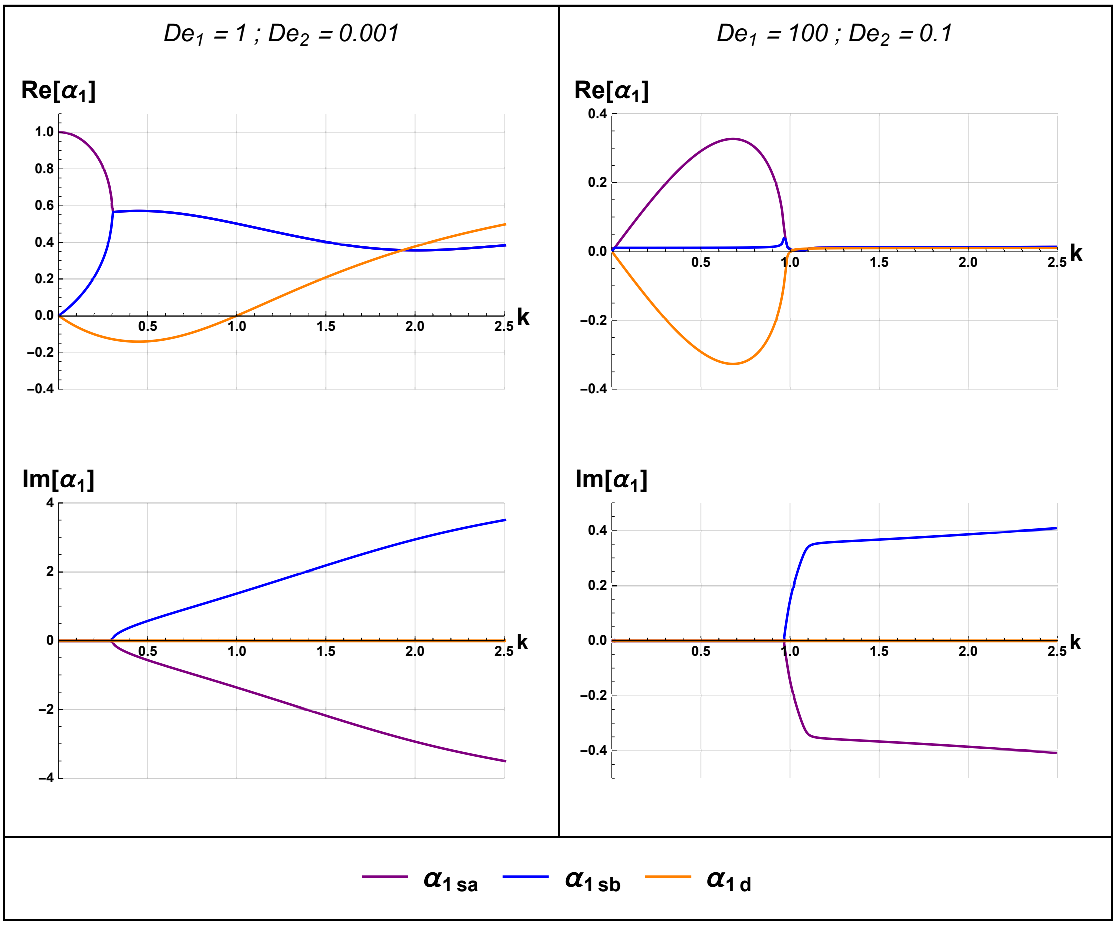

Figure 2. Growth rate and oscillation frequency versus wavenumber k for the three non-trivial first-order modes for $Oh_0 = 0.8$ and different Deborah numbers.

Figure 2 depicts the growth rate and oscillation frequency of the three solutions (α_{1d}, α_{1sa},α_{1sb}) versus the wavenumber for two cases differing by Deborah numbers values but choosen to keep the ratio De_1/De_2 constant. To facilitate the reading of the solutions, we shall define two particular wavenumbers: k_c the cut-off wavenumber of the instability and $k^\star$ the oscillation critical wavenumber. k_c is the wavenumber above which the jet is stable, i.e. all solutions exhibit a positive growth rate (the opposite sign of the one chosen in the formulation of the time dependency in the exponential function). $k^\star$ is the wavenumber above which oscillations exist, i.e. there are solutions with non-identically zero oscillation frequencies. In both cases, it can be seen that $k_c = 1$, the unstable mode is α_{1d}, $0 < k^\star < k_c$ and the value of $k^\star$ is slightly different between the two cases. These features remain true when the Ohnesorge number is varied, except that the dependence of $k^\star$ is stronger. Following the same analysis as the one described in García & González [9], we can distinguish three possible behaviours of the viscoelastic jet for the range of wavenumbers studied here:

- Growth without oscillations: For $k \leq k^\star$, α_{1sa} and α_{1sb} have positive growth rates and α_{1d} has a negative growth rate. The jet is destabilised by α_{1d} the dominant mode eigenvalue. There are no associated oscillations since the oscillation frequencies of all the solutions are identically zero.
- Growth with oscillations: For $k^\star < k < k_c$, α_{1sa} and α_{1sb} are complex conjugates, and α_{1d} remains real negative. There are oscillations from the two sub-dominant mode eigenvalues α_{1sa} and α_{1sb}.
- Damping with oscillations: For $k \geq k_c$, the two solutions α_{1sa} and α_{1sb} are still complex conjugates, but α_{1d} becomes real positive, which indicates the stabilising behaviour of the jet.

In Figure 3, the cut-off wavenumber k_c and the critical wavenumber $k^\star$ are plotted against the Ohnesorge number Oh_0, for $De_1 = 1$ and $De_2 = 0.001$.
We note that the cut-off wavenumber always equals 1. The value of $k^\star$ is less than the cut-off wavenumber for all Ohnesorge number, and decreases by increasing Oh_0. This decrease is less dramatic for large Ohnesorge numbers. Nonetheless, there seems to be either a plateau or a discontinuity around $Oh_0^\star = 0.2$, on which we are still investigating.
From others of our results, not reported here, we observed that decreasing the ratio De_1/De_2 does not impact the values of the cut-off wavenumber, and the critical Ohnesorge number $Oh_0^\star$ seems to have the same value. The

dependency of $k^\star$ on Oh_0 remains the same even though we note that $k^\star$ becomes a bit smaller for $Oh_0 < Oh_0^\star$, and becomes a bit larger for $Oh_0 > Oh_0^\star$.

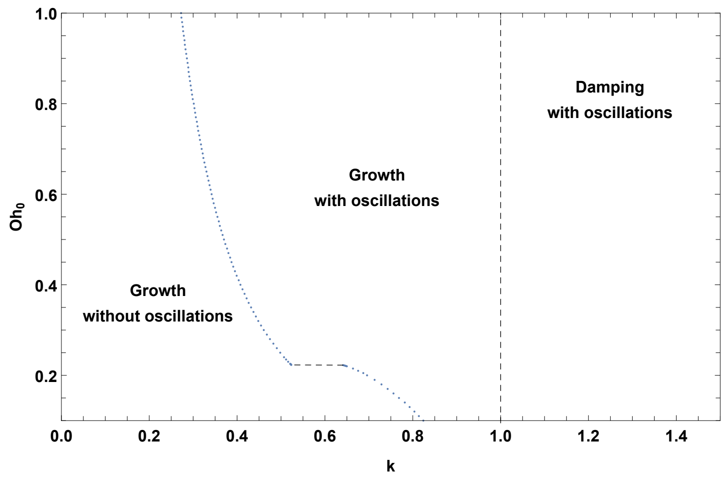

Figure 3. Plateau cut-off wavenumber, critical wavenumber curve, and three possible behaviour of the viscoelatic jet for the range of wavenumbers studied, and for $De_1 = 1$ and $De_2 = 0.001$.

We now formulate the complete expression of the first-order flow properties, taking into account the additional mode, as follows:

$$\begin{aligned}
\eta_1(z,t) &= \sum_{u=d,sa,sb} \hat{\eta}_{1u} e^{-\alpha_{1u}t}\cos(kz) \\
u_1(r,z,t) &= \sum_{u=d,sa,sb} \hat{\eta}_{1u}\left[\left(2k^2 Oh_{v1u} - \alpha_{1u}\right)\frac{I_1(kr)}{I_1(k)} - 2Oh_{v1u}k^2\frac{I_1(l_{v1u}r)}{I_1(l_{v1u})}\right] e^{-\alpha_{1u}t}\cos(kz) \\
w_1(r,z,t) &= -\sum_{u=d,sa,sb} \hat{\eta}_{1u}\left[\left(2k^2 Oh_{v1u} - \alpha_{1u}\right)\frac{I_0(kr)}{I_1(k)} - 2Oh_{v1u}kl_{v1u}\frac{I_0(l_{v1u}r)}{I_1(l_{v1u})}\right] e^{-\alpha_{1u}t}\sin(kz) \\
p_1(r,z,t) &= \sum_{u=d,sa,sb} \frac{\hat{\eta}_{1u}\alpha_{1u}}{k}\frac{I_0(kr)}{I_1(k)}\left(2k^2 Oh_{v1u} - \alpha_{1u}\right) e^{-\alpha_{1u}t}\cos(kz)
\end{aligned}$$

Given an arbitrary value for one of the amplitude coefficients, the two others are obtained from the two initial conditions.

Second-order equations and solutions

The second-order set of equations reads:

$$\begin{aligned}
\frac{1}{r}(ru_2)_{,r} + w_{2,z} &= 0 && \text{for} \quad r \le r_s \\
u_{2,t} + u_1 u_{1,r} + w_1 u_{1,z} &= -p_{2,r} + \frac{1}{r}(r\tau_{rr2})_{,r} - \frac{\tau_{\theta\theta 2}}{r} + \tau_{zr2,z} && \text{for} \quad r \le r_s \\
w_{2,t} + u_1 w_{1,r} + w_1 w_{1,z} &= -p_{2,z} + \frac{1}{r}(r\tau_{rz2})_{,r} + \tau_{zz2,z} && \text{for} \quad r \le r_s \\
\boldsymbol{\tau}_2 + De_1\boldsymbol{\tau}_{2,t} &= 2\,Oh_0(\boldsymbol{D}_2 + De_2\boldsymbol{D}_{2,t}) && \text{for} \quad r \le r_s \\
&\quad - De_1\left[(\boldsymbol{v}_1\cdot\boldsymbol{\nabla})\,\boldsymbol{\tau}_1 + \boldsymbol{W}_1\cdot\boldsymbol{\tau}_1 - \boldsymbol{\tau}_1\cdot\boldsymbol{W}_1 - \boldsymbol{D}_1\cdot\boldsymbol{\tau}_1 - \boldsymbol{\tau}_1\cdot\boldsymbol{D}_1\right] \\
&\quad + 2\,De_2\left[(\boldsymbol{v}_1\cdot\boldsymbol{\nabla})\,\boldsymbol{D}_1 + \boldsymbol{W}_1\cdot\boldsymbol{D}_1 - \boldsymbol{D}_1\cdot\boldsymbol{W}_1 - 2\,\boldsymbol{D}_1\cdot\boldsymbol{D}_1\right] \\
u_2 - u_{1,r}\eta_1 - w_1\eta_{1,z} &= \eta_{2,t} && \text{at} \quad r = 1 \\
\tau_{rz2} + \eta_1\tau_{rz1,r} + \eta_{1,z}(\tau_{rr1} - \tau_{zz1}) &= 0 && \text{at} \quad r = 1 \\
p_2 + \eta_2 + \eta_{2,zz} - \tau_{rr2} &= -\eta_1 p_{1,r} + \eta_1\tau_{rr1,r} - 2\eta_{1,z}\tau_{rz1} + \eta_1^2 - \frac{1}{2}\eta_{1,z}^2 && \text{at} \quad r = 1 \\
\eta_2(z,t=0) &= -\frac{1}{4} \\
\eta_{2,t}(z,t=0) &= 0
\end{aligned}$$

The second-order solutions for the velocity components, pressure, extra-stress and jet surface shape are sought under the forms :

$$\begin{aligned}
u_2(r,z,t) =& u_{21}(r,z,t) + u_{22}(r,z,t)\\
w_2(r,z,t) =& w_{21}(r,z,t) + w_{22}(r,z,t)\\
p_2(r,z,t) =& p_{21}(r,z,t) + p_{22}(r,z,t)\\
\eta_2(r,z,t) =& \eta_{21}(r,z,t) + \eta_{22}(r,z,t)
\end{aligned}$$

where the subscripts "21" denote the contributions of first order through nonlinear terms and where the subscripts "22" correspond to the contributions of second order through linear terms.

The contributions subscripted with "22" are directly deduced from the linear problem, since they are of the same structure as for first order, with the wavenumber k replaced by $2k$, the growth rate α_1 replaced by α_2, the factor β_1 replaced by $\beta_2 = (1-\alpha_2 De_2)/(1-\alpha_2 De_1)$, the modified Ohnesorge number Oh_{v1} replaced by $Oh_{v2} = \beta_2 Oh_0$ and the modified wavenumber l_{v1}^2 replaced by $l_{v2}^2 = 4k^2 - \alpha_2/Oh_{v2}$. The second-order mode eigenvalues are obtained as roots of the dispersion relation (DR) but formulated with the modified parameters exposed before. The second contributions to the second-order solutions are then given by:

$$\begin{aligned}
\eta_{22}(z,t) =& \sum_{u=d,sa,sb} \hat{\eta}_{22u} e^{-\alpha_{2u}t}\cos(2kz)\\
u_{22}(r,z,t) =& \sum_{u=d,sa,sb} \hat{\eta}_{22u}\left[\left(8k^2 Oh_{v1u} - \alpha_{2u}\right)\frac{I_1(kr)}{I_1(k)} - 8Oh_{v2u}k^2\frac{I_1(l_{v2u}r)}{I_1(l_{v2u})}\right]e^{-\alpha_{2u}t}\cos(2kz)\\
w_{22}(r,z,t) =& -\sum_{u=d,sa,sb} \hat{\eta}_{22u}\left[\left(8k^2 Oh_{v2u} - \alpha_{2u}\right)\frac{I_0(kr)}{I_1(k)} - 8Oh_{v2u}kl_{v2u}\frac{I_0(l_{v2u}r)}{I_1(l_{v2u})}\right]e^{-\alpha_{2u}t}\sin(2kz)\\
p_{22}(r,z,t) =& \sum_{u=d,sa,sb} \frac{\hat{\eta}_{22u}\alpha_{2u}}{2k}\frac{I_0(kr)}{I_1(k)}\left(8k^2 Oh_{v2u} - \alpha_{2u}\right)e^{-\alpha_{2u}t}\cos(2kz)
\end{aligned}$$

The amplitudes $\hat{\eta}_{22d}$, $\hat{\eta}_{22sa}$ and $\hat{\eta}_{22sb}$ are derived using the same method as for $\hat{\eta}_{1d}$, $\hat{\eta}_{1sa}$ and $\hat{\eta}_{1sb}$, and therefore depend only on the solutions of the second-order dispersion relation.

Concerning the contributions subscripted with "21", they exhibit exponential time dependencies with the three non-identically zero solutions of the first-order dispersion relation, so that:

$$\begin{aligned}
u_{21}(r,z,t) =& u_{21}^{sa}(r,z)e^{-2\alpha_{sa}t} + u_{21}^{sb}(r,z)e^{-2\alpha_{sb}t} + u_{21}^{d}(r,z)e^{-2\alpha_{d}t}\\
&+ u_{21}^{sa,sb}(r,z)e^{-(\alpha_{sa}+\alpha_{sb})t} + u_{21}^{sa,d}(r,z)e^{-(\alpha_{sa}+\alpha_{d})t} + u_{21}^{sb,d}(r,z)e^{-(\alpha_{sb}+\alpha_{d})t}\\
w_{21}(r,z,t) =& w_{21}^{sa}(r,z)e^{-2\alpha_{sa}t} + w_{21}^{sb}(r,z)e^{-2\alpha_{sb}t} + w_{21}^{d}(r,z)e^{-2\alpha_{d}t}\\
&+ w_{21}^{sa,sb}(r,z)e^{-(\alpha_{sa}+\alpha_{sb})t} + w_{21}^{sa,d}(r,z)e^{-(\alpha_{sa}+\alpha_{d})t} + w_{21}^{sb,d}(r,z)e^{-(\alpha_{sb}+\alpha_{d})t}\\
p_{21}(r,z,t) =& p_{21}^{sa}(r,z)e^{-2\alpha_{sa}t} + p_{21}^{sb}(r,z)e^{-2\alpha_{sb}t} + p_{21}^{d}(r,z)e^{-2\alpha_{d}t}\\
&+ p_{21}^{sa,sb}(r,z)e^{-(\alpha_{sa}+\alpha_{sb})t} + p_{21}^{sa,d}(r,z)e^{-(\alpha_{sa}+\alpha_{d})t} + p_{21}^{sb,d}(r,z)e^{-(\alpha_{sb}+\alpha_{d})t}\\
\boldsymbol{\tau}_{21}(r,z,t) =& \boldsymbol{\tau}_{21}^{sa}(r,z)e^{-2\alpha_{sa}t} + \boldsymbol{\tau}_{21}^{sb}(r,z)e^{-2\alpha_{sb}t} + \boldsymbol{\tau}_{21}^{d}(r,z)e^{-2\alpha_{d}t}\\
&+ \eta_{21}^{sa,sb}(r,z)e^{-(\alpha_{sa}+\alpha_{sb})t} + \eta_{21}^{sa,d}(r,z)e^{-(\alpha_{sa}+\alpha_{d})t} + \eta_{21}^{sb,d}(r,z)e^{-(\alpha_{sb}+\alpha_{d})t}
\end{aligned}$$

Using the continuity equation, we eliminate the second-order velocities from the momentum equations and then we obtain:

$$\Delta p_{21} = -\boldsymbol{\nabla}\cdot[(\boldsymbol{v}_1\cdot\boldsymbol{\nabla})\boldsymbol{v}_1] + \boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_{21}]$$

After using the Lamé identity for the convective derivative of $\boldsymbol{v}_1$ and re-writing the cross product of the first-order velocity $\boldsymbol{v}_1$ with its curl, the previous equation becomes:

$$\Delta[p_{21} + \boldsymbol{v}_1^2/2] = -\boldsymbol{\nabla}\cdot\left[\left(\sum_{u=d,sa,sb}\frac{\alpha_{1u}}{Oh_{v1u}}\frac{\psi_{1u}^b}{r^2}\right)\boldsymbol{\nabla}\psi_1\right] + \boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}\cdot\boldsymbol{\tau}_{21}]$$

where ψ_1 corresponds to the first-order stream function with the property that: $u_1 = -\psi_{1,z}/r$ and $w_1 = \psi_{1,r}/r$. According to the first-order solutions, the expression of ψ_1 is given by:

$$\begin{aligned}
\psi_1(r,z,t) =& \sum_{u=d,sa,sb} -\frac{rI_1(kr)}{kI_1(k)}\hat{\eta}_{1u}(2k^2 Oh_{v1u} - \alpha_{1u})e^{-\alpha_{1u}t}\sin(kz) + \sum_{u=d,sa,sb} 2kOh_{v1u}r\hat{\eta}_{1u}\frac{I_1(l_{v1u}r)}{I_1(l_{v1u})}e^{-\alpha_{1u}t}\sin(kz)\\
:=& \sum_{u=d,sa,sb}\psi_{1u}^a(r,z,t) + \sum_{u=d,sa,sb}\psi_{1u}^b(r,z,t)
\end{aligned}$$

The use of the stream function automatically ensures that the continuity equation is satisfied.

The Poisson equation for the modified pressure $p_{21} + \boldsymbol{v}_1^2/2$ has a similar structure as in the Newtonian case, but in the latter case, the term with the divergence of the extra stress tensor $\boldsymbol{\tau}_2$ on the right-hand side of this equation vanishes. The presence of this term in this equation does not change the methodology of solution of the second-order system compared to the Newtonian one, but requires an additional use of the polynomial approximation of product of modified Bessel functions of first kind with different arguments, already used by Renoult et al. [3].

This weakly nonlinear stability analysis could bring us the first details concerning the effects of the non-linearities on the jet stability characteristics, which arise from the non-planar geometry, the momentum advection, and the RCE.

Conclusion

A weakly nonlinear temporal stability analysis of a viscoelastic liquid jet was performed. The first and second-order sets of governing equations were found to be quite similar to the Newtonian ones, and allowed us to solve the problem by the same methodology as in Renoult et al. [3]. Nonetheless, an additional eigenvalue appears in the non-Newtonian case while finding the roots of the dispersion relation. Taking into account this novelty, the expression of the flow quantities such as surface jet shape, velocity components, and pressure are given. Concerning the second-order solutions, the main difference between the Newtonian and the viscoelastic case is the presence of an additional term in the Poisson equation for the pressure, that actually just requires an additional polynomial approximation of products of Bessel functions of first kind with different arguments. From our results, we can affirm that the viscoelastic liquid jet gets stabilised for wavenumbers larger than unity, whatever are the rheological characteristics of the fluid. Nevertheless, parameters such as the zero-shear viscosity, the relaxation time and the deformation retardation time have notable influence on the values of the eigenvalues. The influence of these parameters and of the three sources of non-linearities are still under investigation.

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